# COMPUTATION OF $\mathrm{QCoh}(\mathrm{B}\mathbb{Z}_\mathrm{p})$

I want to describe how to compute $\mathrm{QCoh}(\mathrm{B}\mathbb{Z}_\mathrm{p})$ over $\mathbb{F}_p$, where $\mathbb{Z}_p$ is the inverse limit of $\mathbb{Z}/p^n$ in the category of affine schemes, and $\mathbb{Z}/p^n$ is the affine scheme representing the Zariski sheafification of the constant presheaf $\mathbb{Z}/p^n$. Concretely, $\mathbb{Z}/p^n$ is $\operatorname{Spec}\operatorname{Fun}(\mathbb{Z}/p^n, \mathbb{F}_p)$ and $\mathbb{Z}_p$ is $\operatorname{Spec}\operatorname{Cont}(\mathbb{Z}_p, \mathbb{F}_p)$–with group scheme structure coming from group structure on the profinite set $\mathbb{Z}_p$.

By examination (or Lurie-Barr-Beck), $\mathrm{QCoh}(\mathrm{B}\mathbb{Z}_\mathrm{p})$ is the category of comodules over the coalgebra $\operatorname{Cont}(\mathbb{Z}_p, \mathbb{F}_p)$. We find this coalgebra/comonad on the dual side as follows. The Fourier-Cartier dual (for category number 0) of $\mathbb{Z}_p$ is $\mathbb{G}_m^\wedge$. Thus we should compare by general theory (because we are studying quasi-coherent sheaves) $\mathrm{QCoh}(\mathrm{B}\mathbb{Z}_\mathrm{p})$ with $\mathrm{QCoh}(\mathbb{G}_\mathrm{m}^\wedge)$, i.e. the quasicoherent sheaves on the formal scheme of $\mathbb{G}_m$ completed at 1. We will first think of this as subcategory of $\mathrm{QCoh}(\mathbb{G}_\mathrm{m})$ supported at 1 and denote it as $\Gamma_*\mathrm{QCoh}(\mathbb{G}_\mathrm{m})$– these categories are abstractly the same by Greenlees-May duality.

The composition

$$\Gamma_*\mathrm{QCoh}(\mathbb{G}_\mathrm{m}) \to \mathrm{QCoh}(\mathbb{G}_\mathrm{m}) \xrightarrow{\pi_*} \mathrm{QCoh}(*)$$

where the first functor is inclusion of torsion subcategory and $*$ represents the final object of affine schemes over $\mathbb{F}_p$, namely $\operatorname{Spec}\mathbb{F}_p$, is left adjoint to $\Gamma_*\pi^\times$–where $\pi^\times$ denotes the right adjoint to $\pi_*$. General theory tells us that

$$\Gamma_*\pi^\times(\_) \cong \Gamma_*\pi^!(\_) \cong \pi^*(\_) \otimes \Gamma_*\omega_{\mathbb{G}_m} \cong \pi^*(\_) \otimes E$$

where $E$ is the injective hull of the residue field of $\mathbb{G}_m$ at 1. Hence $\Gamma_*\pi^\times(\mathbb{F}_p)$ is discrete, and we can compute it as the subspace of

$$\operatorname{Hom}(O(\mathbb{G}_m), \mathbb{F}_p)$$

on which the function $t-1$ acts nilpotently (where $t$ is the pullback of the canonical function on $\mathbb{A}^1$ to $\mathbb{G}_m$). In other words,

$$\Gamma_*\pi^\times(\mathbb{F}_p) \cong \operatorname{colim}_n O(\mu_{p^n})^\vee \cong \operatorname{colim}_n O(\mathbb{Z}/p^n) \cong \operatorname{Cont}(\mathbb{Z}_p)$$

with comonad structure given by group structure on $\mathbb{Z}_p$ by examination.

Now as $\Gamma_*\pi_*$ is comonadic (it's a composition of two comonadic functors), we have the equivalence

$$\mathrm{QCoh}(\mathrm{B}\mathbb{Z}_\mathrm{p}) \cong \Gamma_*\mathrm{QCoh}(\mathbb{G}_\mathrm{m})$$

(where tensor product on left corresponds to convolution on right–we omit this proof but it's an easy check). Hence $B\mathbb{Z}_p$ is a perfect stack over $\mathbb{F}_p$ and the structure sheaf generates. Therefore

$$\mathrm{QCoh}(\mathrm{B}\mathbb{Z}_\mathrm{p}) \cong \operatorname{Mod} H^*(S^1, \mathbb{F}_p)$$

where on the last step we refer to p-adic homotopy theory (as cohomology over $\mathbb{F}_p$ factors through profinite completion).